MID-SEMESTRAL EXAMINATION
B.MATH.HONS.Ist year
ALGEBRA I
SEMESTER - I : 2000-2001

• *Attempt TWO questions from the first five and SIX questions from the rest.*

Q 1: 12 Marks.
Find the characteristic polynomial of the matrix $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbf{C})$ and verify that A is a root of this polynomial.

Q 2: 12 Marks.
Let $A \in M_n(\mathbf{C})$ be skew-symmetric i.e., $A^t = -A$. If A is invertible, show that n must be even.

Q 3: 10 Marks.
Suppose a, b, c are distinct complex numbers. Prove that the system of linear equations $Ax = B$ has a solution, where $A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Q 4: 12 Marks.
Show that for each natural number n and $A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in M_2(\mathbf{C})$, there exists $B \in GL_2(\mathbf{C})$ such that $BAB^{-1} = A^{n^2}$.

Q 5: 12 Marks.
Let $U = \{M \in M_n(\mathbf{C}) : M$ is upper tiangular, $m_{ii} = 1 \forall i\}$. If $A, B \in U$, prove that $C = ABA^{-1}B^{-1} \in U$ and also that $c_{i,i+1} = 0 \forall i = 1, \cdots, n-1$.

Q 6: 15 Marks.
Let G be any group.
(i) If $g \in G$ has order n, find the order of g^d.
(ii) If $a, b \in G$, prove that $O(ab) = O(ba)$.
(iii) If $a^3 = e$ and $aba^{-1} = b^2 \neq e$, find $O(b)$.

Q 7: 10 Marks.
Let $\theta : G \to H$ be an onto homomorphism of groups. Suppose N is a normal subgroup of G. Prove that $\theta(N)$ is a normal subgroup of H.

Q 8: 15 Marks.
If G is a group which has only finitely many subgroups in all, show that G must be finite.

Q 9: 12 Marks.
Let G be an abelian group of order n. If $(m, n) = 1$, prove that the map $\theta : G \to G; g \mapsto g^m$ is an isomorphism.

Q 10: 10 Marks.
Suppose G is any group (not necessarily finite) and H is a subgroup such that there are exactly n distinct left cosets of H in G. Prove that n is also the number of distinct right cosets of H in G.

Q 11: 14 Marks.
Let $G = S_4$ and $N = \{I, (12)(34), (13(24), (14)(23)\}$. Prove that N is normal in G and $G/N \cong S_3$.

Q 12: 12 Marks.
Prove that the number of group homomorphisms from $\mathbf{Z}_m$ to $\mathbf{Z}_n$ is (m, n).

Q 13: 10 Marks.
Let G_1, G_2 be groups and let $g_1 \in G_1, g_2 \in G_2$ have orders n_1, n_2 respectively. Find the order of $(g_1, g_2) \in G_1 \times G_2$.

Q 14: 10 Marks.
Let $K \leq H \leq G$ with G finite. Prove that

$$[G : K] = [G : H][H : K]$$

Q 15: 15 Marks.
Let $\sigma \in S_n$ have order a prime p. Then,show that

$$\#\{i \leq n : \sigma(i) = i\} \equiv n \quad mod \ p$$